M-STRUCTURE AND INTERSECTION PROPERTIES OF BALLS IN BANACH SPACES

BY

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ABSTRACT

This paper contains a complex version of Theorem 5.4 (Dominated Extension Theorem) from the paper "Structure in real Banach spaces" by E. Effros and the present author (to appear in Annals of Mathematics).

The lecture given at the Symposium, was a survey covering some of the material in the forthcoming papers by E. Effros and the present author [3], [4] together with some related recent results by various authors [6], [11], [17], [19]. Almost all of this will be published in the papers listed above. The only major exception is a complex version of Theorem 5.4 of [1] (The Dominated Extension Theorem), which will be stated and proved in this note. The method of proof is similar to that of Theorem 4.5 of [5] (which is actually generalized by the present theorem).

A subspace J of a real Banach space V is said to be an *M-summand* if there is a subspace H of V such that $J \cap H = \{0\}$, $J + H = V$, and for each $j \in J$, $h \in H$

$$\|j + h\| = \max\{\|j\|, \|h\|\}. \tag{1}$$

Similarly a subspace N of V is said to be an *L-summand* if there is a subspace M of V such that $N \cap M = \{0\}$, $N + M = V$, and for each $p \in N$, $q \in M$

$$\|p + q\| = \|p\| + \|q\|. \tag{2}$$

It is easily verified that if J is an M-summand of V, then its annihilator J^0 is an L-summand of V^*. But these annihilators will not exhaust the class of all w^*-closed L-summands in V^*, in general, although it will be so for reflexive spaces. (See Cor. 2.6 and Cor. 2.9 of [4])

It is also easily verified that the annihilator N^0 of an L-summand N of V will be an M-summand of V^* (Prop. 2.5 (b) of [4]). The question whether these annihilators will exhaust the class of all w^*-closed M-summands in V^*, was left open in [4] (Problem 1, §7), and it was later answered affirmatively by Cunningham et al. [11].

Following [3] we shall use the term *M-ideal* to denote a closed subspace J of V for which J^0 is an L-summand of V^*. By the above remarks the M-ideals include but are more extensive than the M-summands. Likewise we may define an *L-ideal* to be a closed subspace N of V for which N^0 is an M-summand of V^*, but it follows from the above remarks that these L-ideals will be exactly the L-summands. Hence we may (and shall) use the words *L-ideal* and *L-summand* interchangeably.

If N is an L-summand, then the corresponding subspace M involved in the definition is unique, and the projection onto N determined by $V = N \oplus M$ will be called the *L-projection* corresponding to N. The concept of an L-projection was defined in 1960 by Cunningham [9] who proved that *any two L- projections will commute.* (See also [3, §3].) Similar results hold for M-summands and the corresponding "M-projections", but they will not be needed in the sequel since we shall be interested mainly in the M-ideals J in V and the corresponding L-ideals (or L-summands) $N = J^0$ in V^*.

At this point we pass to a complex Banach space V with subordinate real space V_r. *If N is an L-ideal of V_r, then N will be closed under multiplication by complex scalars, and the corresponding L-projection will be a complex-linear operator.* (The proof makes use of the commutativity of L-projections. See §2 of [19].) Applying this theorem to V^*, one gets a similar result for M-ideals. Specifically, *if M is an M-ideal of V_r, then M is a complex-linear subspace of V.*

Thus, if we define *L- and M-ideals* in a *complex Banach space V* by the same formal requirements as in a real space, then the requirement to complex linearity will not limit the supply of such ideals. They are (apart from the different field of scalars) exactly the same as the corresponding real ideals in V_r.

To fix the ideas we shall identify the M-ideals in a few important particular cases:

1) If V is the space $A(K)$ of continuous affine real valued functions on a compact convex subset K of a locally convex Hausdorff space, then the M-ideals of V are exactly the annihilators of the split-faces of K. (See Theorem 6.10 of [4] and Prop. II. 6.23 of [1] for the relevant information on split faces.)

2) If V is a Lindenstrauss space (i.e. if V^* is an L^1-space), then the M-ideals of V are exactly the annihilators of the bifaces of the closed unit ball of V^* (See §8 of [4], and see [16] for the definition and basic properties of bifaces. See also [17] for the complex case.)

3) If V is a C^*-algebra, then the M-ideals of V are exactly the 2-sided norm closed ideals. If V is weakly closed, i.e., a von Neumann algebra, then the M-summands are the weakly closed 2-sided ideals. (See Prop. 6.16 and 6.18 of [4].)

4) If V is a uniform algebra (i.e. a norm closed subalgebra of $C(X)$ for some metrizable compact Hausdorff space X), then the M-ideals of V are exactly the ideals of the form $\{f \in V \mid f \equiv 0 \text{ on } E\}$ where $E \subseteq X$ is a peak set for V. The M-summands correspond in the same way to open-closed peak sets $E \subseteq X$. (These results are due to Hirsberg [19].)

In [4] it is shown how the M-ideals may be used to study a Banach space in much the same way that two sided ideals are used in ring theory. There is defined a "structure space" consisting of "primitive ideals" topologized by a "hull-kernel" operation, and it is shown how this will reduce to known structure spaces in the special cases mentioned above (See also [19]). Also it generalizes the structure space methods used in connection with elliptic and parabolic boundary value problems by Effros and Kazdan [18]. A main result of [4] is that the bounded continuous functions on Prim V naturally correspond to certain geometrically determined linear operators on V. (They are the "M-bounded operators" or "multipliers" on V. See Theorems 4.8 and 4.9 of [4].) This gives a generalization of the Dauns-Hofmann Theorem for C^*-algebras [12]. (See also [13].)

We shall not pursue this line of investigation in the present note. Instead we turn to the elementary, but non-trivial, problem to characterize the M-ideals of V intrinsically, i.e., without use of V^*. In this connection we shall need intersection properties of subspaces and balls.

We say that a subspace J of V enjoys then *n-ball property* if for any collection $B_1, \cdots, B_n$ of n open balls such that $B_i \cap J \neq \emptyset$ for $i = 1, \cdots, n$ and $B_1 \cap \cdots \cap B_n \neq \emptyset$, necessarily $B_1 \cap \cdots \cap B_n \cap J \neq \emptyset$.

If J enjoys the n-ball property for some n, then it will clearly also enjoy the m-ball property for any $m < n$. The passage to larger values $(m > n)$ is less trivial. But it can be proved that *the 3-ball property is sufficient to ensure the n-ball property for $n \geqq 3$, and that the 2-ball property will not suffice.* (Theorem 5.9 of [3]).

It can also be proved that the n-ball property is equivalent to the corresponding property for n *closed* balls provided their interiors have a non-empty intersection. (These interiors need not have non-empty intersection with J though. For details see the equivalence of (b) and (c) in Theorem 5.8 of [3]. For the irredundancy of the condition on the interiors, see also Remark 5.10 of [3] which is based on an example by Stefánson who made an essentially equivalent observation in connection with a problem of Dixmier concerning ideals in a von Neumann algebra [23].)

The paper [3] is largely devoted to a proof that *a closed subspace of V is an M-ideal if and only if it enjoys the* 3-*ball property.*

This result is proved by way of a number of intermediate characterizations, some of which will undoubtedly be more useful than the 3-ball property itself. We shall not give the details in this note, but only point out that the measure condition of Theorem 4.5 of [3] has proved useful for the identification of M-ideals in specific cases (e.g., for uniform algebras [19]), and that the Dominated Extension Property of Theorem 5.4 may have interesting analytical applications (See [5]). The latter result is stated and proved for real spaces in [3], and it leans heavily on the linear ordering of the scalar field $\mathbb{R}$.

In the remaining part of this note we shall establish a complex analoueg of Theorem 5.4 of [3]. The proof of this result is based on known techniques and is perhaps less interessting than some of the proofs which have been bypassed in the above presentation. However, since we can not give a satisfactory reference, we shall give the proof. It is based directly on the definition of an L-ideal, and it makes use of one technical result from [3] (Lemma 4.1).

In the sequel we shall consider a fixed complex Banach space V with dual space W, and we shall use the letter K to denote the closed unit ball of W endowed with the w^*-topology. We shall also use the symbol $V(K)$ to denote the canonical image of V in $W^* = V^{**}$ regarded as a (closed) subspace of $C(K)$. If $v \in V$ and $p \in K$ then we shall denote the value of p at v by $v(p)$ as well as $p(v)$. Finally we shall use the symbol $V(K)^{\perp}$ to denote the space of all complex measures on the w^*-compact set K which vanish on $V(K)$.

We can now state the result:

THEOREM. *Suppose that N is a w^*-closed L-ideal in W. Let $v_0 \in V$, and let g be a w^*-lower-semicontinuous concave real valued function on K. If*

$$g(p) > 0 \qquad \text{for all } p \in K, \tag{3}$$

and

$$(4) \qquad |v_0(p)| \leqq g(p) \quad \textit{for all } p \in K \cap N,$$

then there exists a $\bar{v} \in V$ *such that* $\bar{v} = v_0$ *on* N *and*

$$(5) \qquad |\bar{v}(p)| \leqq g(p) \quad \textit{for all } p \in K.$$

What makes this type of theorem more complicated than ordinary Hahn-Banach extensions, is the combination of a uniform bound (5) and weak *continuity of the extended function. Without any additional requirement on the w^*-closed space N, the above theorem would not prevail; not even with $g = \text{const.}$, since $v_0 | g$ may then fail to have a strictly norm-preserving w^*-continuous extension to all of W.

Note also that if g is *symmetric*, i.e., $g(p) = g(-p)$ for all $p \in K$, then we may replace (3) by the condition

$$(3) \qquad \check{g}(0) > 0$$

which is used in Theorem 5.4 of [3]. (Compare with the customary passage from the real to the complex Hahn-Banach Theorem where a symmetry condition must be imposed on the bound to permit domination in absolute value.)

The proof of the theorem proceeds in three steps. First we construct explicitly a "weak" solution which satisfies (5) but fails to be w^*-continuous. Then we use this weak solution to show that there exist w^*-continuous extensions of $v_0|_N$ which satisfy (5) up to an ε. Finally we use an inductive argument to show that we can take $\varepsilon = 0$.

Let e be the L-projection corresponding to N and define for every $v \in V$ the function e^*V on W by the usual convention

$$(6) \qquad (e^*v)(p) = v(ep) \quad \text{for all } p \in W.$$

Clearly e^*v is linear. In the sequel we shall only be interested in the restriction of e^*v to K, and for brevity we shall denote this restriction also by e^*v. By application of Lemma 4.1 of [3] to the real and imaginary parts of v, we conclude that e^*v is a linear combination of w^*-semi-continuous functions on K. (The proof of Lemma 4.1 is an elementary but somewhat technical approximation argument based on the definition of an L-ideal.)

LEMMA 1. *With the assumptions of the theorem we have*

$$(7) \qquad |(e^*v_0)(p)| \leqq g(p) \quad \textit{for all } p \in K,$$

and the function e^*v_0 is "affiliated with $V(K)$" in the sense that

$$\mu(e^*v_0) = 0 \quad \textit{for all} \quad \mu \in V(K)^{\perp}. \tag{8}$$

PROOF. Clearly it suffices to prove (7) if $\| p \| = 1$. Writing $p_1 = ep$ and $p_2 = p - ep$, we have

$$p = p_1 + p_2, \qquad \| p_1 \| + \| p_2 \| = \| p \| = 1$$

If $p_1 = 0$ or $p_2 = 0$ there is nothing to prove. Otherwise let $\lambda_i = \| p_i \|$, $q_i = \lambda_i^{-1} p_i$ for $i = 1,2$. Then

$$p = \lambda_1 q_1 + \lambda_2 q_2, \qquad \lambda_1 + \lambda_2 = 1,$$

$\lambda_1, \lambda_2 > 0$, $q_1, q_2 \in K$, $e(q_1) = q_1$ and $e(q_2) = 0$. Since g is concave, we get

$$\begin{aligned} |(e^*v_0)(p)| = |v_0(p_1)| = |\lambda_1 v_0(q_1)| &\leqq \lambda_1 g(q_1) \\ &\leqq \lambda_1 g(q_1) + \lambda_2 g(q_2) \leqq g(p). \end{aligned}$$

To prove (8) we consider an arbitrary $\mu \in V(K)^{\perp}$, and we decompose $\mu = \sum_{i=1}^4 \alpha_i \mu_i$ where μ_i are probability measures and α_i are complex coefficients for $i = 1,2,3,4$. We denote the barycenters of μ_i by p_i, and we claim that

$$\sum_{i=1}^{4} \alpha_i p_i = 0. \tag{9}$$

In fact, if $v \in V(K)$ then it follows by (complex) linearity of v and by the assumption $\mu \in V(K)^{\perp}$ that

$$v\left(\sum_{i=1}^{4} \alpha_i p_i\right) = \sum_{i=1}^{4} \alpha_i v(p_i) = \sum_{i=1}^{4} \alpha_i \mu_i(v) = \mu(v) = 0.$$

Since $v \in V$ was arbitrary, this proves (9).

At this point we reproduce an argument from [3]. Since e^*v_0 is a linear combination of w^*-semi-continuous functions on K, it follows from a theorem which goes back to Osgood (see [20] and Ch. IV, §6, no. 2, ex. 9 of [7]) that e^*v_0 is *w^*-quasi-continuous*, i.e., for each compact set $D \subseteq K$ the restriction of e^*v_0 to D has a dense set of points of continuity in D. Choquet proved that any real valued quasi-continuous affine function on a compact convex set will be bounded and satisfy the barycentric calculus [8]. (See also Ch. I, §2 of [1].) Decomposing into real and imaginary components, we transfer this result to complex valued functions. (Recall that the barycentric calculus only deals with *real* measures, in fact, with probability measures.) In particular we get

$$\mu_i(e^*v_0) = (e^*v_0)(p_i), \qquad \text{for } i = 1,2,3,4.$$

Hence

$$\begin{aligned}\mu(e^*v_0) &= \sum_{i=1}^{4} \alpha_i\mu_i(e^*v_0) \\ &= \sum_{i=1}^{4} \alpha_i(e^*v_0)(p_i) = (e^*v_0)\left(\sum_{i=1}^{4} \alpha_i p_i\right) = 0,\end{aligned}$$

and formula (8) is proved.

LEMMA 2. *Again we assume the conditions of the theorem, and we let* $\varepsilon > 0$ *be arbitrary. Then there exists a* $v \in V$ *such that* $v = v_0$ *on* N, *and*

$$|v(p)| < g(p) + \varepsilon \qquad \text{for all } p \in K . \tag{10}$$

PROOF. We write $\phi(p) = g(p) + \varepsilon$ and define a new norm on the space $C(K)$ of w^*-continuous complex functions on K by

$$\|f\|_\phi = \sup_{p\in K} \left|\frac{f(p)}{\phi(p)}\right| \tag{11}$$

Without lack of generality, we can assume g bounded. Then the new norm will be topologically equivalent with the usual sup-norm, and in the dual space $M(K) = C(K)^*$ we shall get the new (dual) norm

$$\|\mu\|_\phi = \|\mu \cdot \phi\|, \tag{12}$$

where $(\mu \cdot \phi)(f) = \mu(\phi f)$ as usual.

Let $U \subseteq C(K)$ be the open unit ball in the new norm, and let a subspace H of $C(K)$ be defined by

$$H = \{v \in V(K) \,|\, v = 0 \text{ on } N \cap K\}. \tag{13}$$

Now we assume for contradiction that

$$(H + v_0) \cap U = \varnothing. \tag{14}$$

By Hahn-Banach, one can extend $H + v_0$ to a closed hyperplane with the open unit ball U on one side. In other words, there exists a $\mu \in C(K)^* = M(K)$ such that μ vanishes on H and

$$\mu(v_0) = 1, \qquad \|\mu\|_\phi = \|\mu \cdot \phi\| \leqq 1. \tag{15}$$

If $v, v' \in V$ and $v = v'$ on N, then $v - v' \in H$, and so $\mu(v) = \mu(v')$. Hence there is a well defined linear functional ξ on $V(K)|_{K \cap N}$ such that

$$\xi(v|_{K \cap N}) = \mu(v), \qquad \text{for all } v \in V(K).$$

Extending ξ to a linear functional on $C(K \cap N)$ by Hahn-Banach, we obtain a measure ν with support in $K \cap N$ such that

$$\nu(v) = \int_{K \cap N} v d\nu = \mu(v), \qquad \text{for all } v \in V(K). \tag{16}$$

From this it follows that $\mu - \nu \in V(K)^{\perp}$. By virtue of (8) we get

$$\mu(e^* v_0) = \nu(e^* v_0). \tag{17}$$

Using the left hand side of (15) together with (16) and remembering that $e^* v_0 = v_0$ on N, we get

$$1 = \mu(v_0) = \int_{K \cap N} (e^* v_0) d\nu = \nu(e^* v_0) \,.$$

By virtue of (7) $|e^* v_0| \leqq g = \phi - \varepsilon$ on K, and by the right hand side of (15)

$$\begin{aligned} |\mu(e^* v_0)| &\leqq \int_K |e^* v_0| \, d|\mu| \\ &\leqq \int_K \phi d|\mu| - \varepsilon|\mu|(K) \leqq 1 - \varepsilon\,|\mu|(K). \end{aligned}$$

The last two formulas contradict (17) and the proof of Lemma 2 is complete.

It will be convenient to apply Lemma 2 in the following modified form which follows when the original lemma is applied with $g = \psi - \varepsilon$ for some sufficiently small ε (which will exist by the semi-continuity).

LEMMA 2′. *Let N and v_0 be as in the Theorem, and let ψ have the same properties as g, but replace* (3) *by the stronger condition*

$$|v_0(p)| < \psi(p) \qquad \text{for all } p \in K \cap N. \tag{18}$$

Then there exists a $v \in V$ such that $v = v_0$ on N and

$$|v(p)| < \psi(p) \qquad \textit{for all } p \in K. \tag{19}$$

PROOF OF THE THEOREM. Applying Lemma 2′ with $v_0/2$ place of v_0 and ψ_1 in place of ψ_2, we obtain a $v_1 \in V$ such that

$$v_1 = \frac{1}{2} v_0 \text{ on } N, \quad |v_1(p)| < \psi_1(p) \text{ for all } p \in K.$$

Next we define a w^*-lower semi-continuous concave function ψ_2 on K by

$$\psi_2 = \left(\frac{1}{2}g\right) \wedge (g - |v_1|).$$

Now $0 < \psi_2 \leqq g$ and $|v_1| = |\frac{1}{2}v_0| \leqq \frac{1}{2}g$ on $K \cap N$. Hence $g - |v_1| \geqq \frac{1}{2}g$ on $K \cap N$, and so $\frac{1}{2}g = \psi_2$ on $K \cap N$.

In the next step we apply Lemma 2′ with $|\frac{1}{2}v_0|$ in place of v_0 and, ψ_2 in place of ψ, which is possible since $|\frac{1}{4}v_0| < \frac{1}{2}g \leqq \psi_2$ on $K \cap N$. Thus we obtain a $v_2 \in V$ such that

$$v_2 = \frac{v_0}{4} \text{ on } N, \qquad |v_2(p)| < \psi_2(p) \text{ for all } p \in K.$$

Next we define

$$\psi_3 = \left(\frac{1}{4}g\right) \wedge [g - (|v_1| + |v_2|)] .$$

Now $0 < \psi_3 \leqq g$ and $|v_1| + |v_2| = \frac{3}{4}|v_0| \leqq \frac{3}{4}g$ on $K \cap N$. Hence

$g - (|v_1| + |v_2|) \geqq \frac{1}{4}g$ on $K \cap N$, and so $\frac{1}{4}g = \psi_3$ on $K \cap N$.

In the next step we apply Lemma 2′ with $\frac{1}{8}v_0$ in the place of v_0; and we continue by induction to construct a sequence $\{v_n\}$ from V such that

$$v_n = 2^{-n}v_0 \text{ on } N, \quad |v_n(p)| < \psi_n(p) \quad \text{for all } p \in K, \tag{20}$$

where

$$\psi_n = (2^{-n+1}g) \wedge \left(g - \sum_{j=1}^{n-1} |v_j|\right). \tag{21}$$

From this we conclude that $|v_n| < g - \Sigma_{j=1}^{n-1}|v_j|$ on K, and so

$$\sum_{j \leqq 1}^{n} |v_j(p)| < g(p) \qquad \text{for all } p \in K, \ n = 1, 2, \cdots .$$

Since g was supposed to be bounded, we get absolute and uniform convergence, and the sum

$$\bar{v} = \sum_{j=1}^{\infty} v_j$$

will be in $V(K)$ and satisfy the requirements

$$\bar{v} = v_0 \text{ on } N, \quad |\bar{v}(p)| \leqq g(p) \quad \text{for all } p \in K.$$

The proof is complete.

REMARK. The theorem of this note applies equally well to real spaces, but an additional argument is necessary to obtain the corresponding theorem of [3] (Theorem 5.4), since in the latter there are no absolute values and condition (3) is replaced by the weaker condition (3)′. Specifically, one may use the Hahn-Banach argument of Lemma 5.1 of [3] to extend the zero functional on N to a w^*-continuous linear functional w on $W = V^*$ such that $w(p) < g(p)$ for all $p \in K$. Then one may apply the theorem of this note to produce a $v \in V$ such that $v \leqq g - w$ on K, and, finally, one writes $\bar{v} = v + w$.

We close with an open problem. Andersen [6] has used tensor products of compact convex sets and their associated $A(K)$-spaces to generalize the Dominated Extension Theorem for split faces to functions with values in a Banach space with the approximation property. It would be of interest to know if the theorem of this note could be generalized in a similar way.

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